Non-linear Physics

Timothy Head
tim.head@student.manchester.ac.uk

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1 Introduction

Recommended books

- G.L. Baker and J.P.Gollub, "Chaotic dynamics: an introduction"
- D.W. Jordan and P. Smith, "Nonlinear ordinary differential equations"

2 General features of dynamical systems

2.1 General form of the equations

Examples of dynamical systems.

Classical dynamics Almost all of mechanics is non-linear, maybe except for the pendulum. $m_i \frac{d^2\vec{r}}{dt^2} = F(\vec{r_1},\ \vec{r_2},\ \dots,\ \frac{d\vec{r_1}}{dt},\ \frac{d\vec{r_2}}{dt})$ is the general form of the equation governing this. We can introduce $\vec{p_i} = m_i\vec{v_i}$ as a new variable. This gives us a set of first-order equations $\frac{d\vec{p_i}}{dt} = \vec{F_i}(...)$.

Pendulum is a good example of this. $ml\frac{d^2\theta}{dt^2} = -mg\sin(\theta) - bl\dot{\theta}$. When we introduce $\omega = \dot{\theta}$ we can rewrite this as two first order equations.

Another example would be two species competing for resources. N_1 is the number of individuals of species 1 and N_2 number of species 2. We can describe that by $\frac{dN_1}{dt} = rN_1$, the growth rate is related to the number of individuals in species 1 at time t=0. r is the growth rate of the species. This is very naive. A more realistic approach is $\frac{dN_1}{dt} = (r_1^{(0)} - a_1N_1 - a_{12}N_2)N_1$similarly $\frac{dN_2}{dt} = (r_2^{(0)} - a_1N_1 - a_{12}N_2)N_2$.

All these equations have the general form of

$$\begin{aligned}\frac{dx_1}{dt} &= f(x_1,\ x_2,\ \dots,\ x_n)\\ \frac{dx_2}{dt} &= f(x_1,\ x_2,\ \dots,\ x_n)\\ \frac{dx_n}{dt} &= f(x_1,\ x_2,\ \dots,\ x_n)\end{aligned}$$

this is a set of coupled linear differential equations.

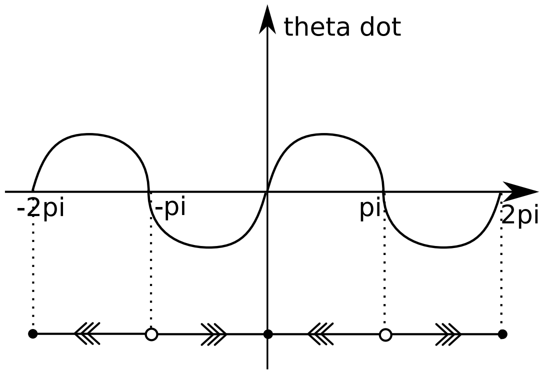

Figure 1: Between 0 and 2π, $\dot{\theta}$ is negative. Lower arrows indicate direction of movement of the particle. At 2π we are at an unstable point, whereas π is a stable point.

Special case is a Hamiltonian system. These are also known as non-dissipative or conservative systems. They only exist for even numbers of n since half the variables represent coordinates and half represent momenta. Let $x_1, x_2, \ldots, x_m$ be $q_1,\ q_2,\ \ldots\ ,\ q_m$be coordinates and $x_{m+1}, x_{m+2},\ \ldots\ , x_n$ be $p_1,\ p_2,\ \ldots\ ,\ p_m$ momenta(where $2m = n$). Then

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \tag{1}$$

and

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \tag{2}$$

where H is the Hamiltonian. Example is a particle moving in 3 dimensions in a potential $V(x,\ y,\ z)$, here $H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(q_1, q_2, q_3)$. Using Eqn 1 and 2 we can find force and velocity of the particle. You will find that all we get is force equals mass times acceleration. We will not look at Hamiltonians at all as they are linear/boring.

2.2 Fixed points in systems with one degree of freedom

Consider a system describe by the following equation

$$ml\frac{d^2\theta}{dt^2} + bl\frac{d\theta}{dt} + mg\sin\theta = 0 \tag{3}$$

in the over-damped limit the first term of this equation can be neglected(imagine a pendulum in trickle). In order to get rid of all the constants we introduce a new time scale $\tau = \frac{t}{T}$ where $T = \frac{bl}{mg}$. This turns equation 3 into $\frac{d\theta}{dt} + \sin\theta = 0$. We can solve this by integration but thats not very helpful or useful. Using a graphical analysis of the original equation which shows the general behavior clearly is much more helpful in understanding the system.

In figure 1 all points for which $\dot{\theta} = 0$ are called fixed points. There are two types of fixed points in this problem, filled dots are *attractors*(or sinks) because

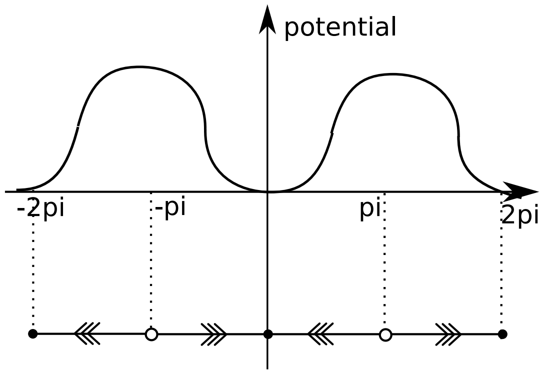

Figure 2: This shows the potential of a pendulum described by equation 3. As in figure 1 filled dots represent attractors and empty circles show repellors.

motion is towards them, empty dots are *repellors*(or sources) because motion is away from them.

Consider starting at $\theta_0 = -\frac{2\pi}{3}$, then $\dot{\theta}$ increases until $\theta = -\frac{\pi}{2}$ then decrease until it reaches zero at $\theta = 0$. Look at figure 1.

Another way to visualise the dynamics of the system is to work in terms of the potential energy. For the previous problem of the pendulum the equation of motion may be written as $ml\ddot{\theta} = \frac{-1}{l}\frac{dV}{dx} - bl\dot{\theta}$ where V is the potential energy. Potential energy of pendulum is $V = -mgl\cos\theta + mgl$.

In the over-damped case(using a scaled potential to get rid of constants)

$$\frac{d\theta}{d\tau} = -\frac{dU}{d\theta}$$

where $U(\theta) = \frac{V(\theta)}{mgl} = 1 - \cos\theta$.

In figure 2 attractors are stable fixed points(minima of potential), repellors are unstable fixed points(maxima of potential). If a potential exists you can think in terms of a mechanical analogy with a ball moving in the potential U. However as the motion is over-damped you need to keep in mind that there is a strong frictional force, so the ball will not “overshoot” the minima as one would expect, it will just come to rest there. Mathematically this is expressed as $\frac{dU}{d\tau} = \frac{dU}{d\theta}\frac{d\theta}{d\tau} = -(\frac{dU}{d\theta})^2 \leq 0$. So U always decreases with time unless we have reached a fixed point.

Linear stability analysis is a general way of investigating the stability of a fixed point. First one has to find a fixed point

$$\frac{d\theta}{d\tau} = -\sin\theta$$

for a fixed point this will be equal to zero, which means θ not changing anymore. Here that would be at $\theta = n\pi$ $n = 0, \pm1, \pm2, ...$. We will use θ^* to denote a fixed point. Write $\theta(\tau) = \theta^* + \hat{\theta}(\tau)$, where $\hat{\theta}$ represents small quantities.

$$\sin\theta = \sin(\theta^* + \hat{\theta}) = \sin\theta^* + \hat{\theta}\sin\theta^* + \hat{\theta}^2 + ...$$

here the first term on RHS will be zero(thats how we chose the fix point). We can now write it as

$$\frac{d\hat{\theta}}{d\tau} = -\cos\theta^* \hat{\theta} = -(-1)^n \hat{\theta}$$

suppose that n is odd $\frac{d\hat{\theta}}{d\tau} = \hat{\theta} \Rightarrow \hat{\theta}(\tau) = A\exp(\tau) = \hat{\theta}(0)\exp(\tau)$.If the start is very near to the fixed point we move away exponentially fast, this must be an unstable fixed point. For even n we get $\frac{d\hat{\theta}}{d\tau} = -\hat{\theta} \Rightarrow \hat{\theta}(\tau) = B\exp(-\tau)$. Here we move towards the fixed point as time increases, this is a stable point(although it does take exponentially long to get there).

For a general first order equation

$$\dot{x} = f(x) \tag{4}$$

fixed points are $f(x^*) = 0$. Consider a particular fixed point x^*, so we can write $x(t) = x^* + \hat{x}(t)$. This gives $\dot{x} = \dot{\hat{x}}$. Using this to expand equation 4 about a fixed point

$$f(x) = f(x^* + \hat{x}) = f(x^*) + \hat{x}f'(x)|_{x=x^*} + ...$$

this gives $\frac{d\hat{x}}{dt} = \hat{x}f'(x^*)$. If $f'(x^*) > 0$ the fixed point is unstable, if $f'(x^*) < 0$ the fixed point is stable. If $f'(x^*) = 0$ we would need to keep next order in f(x) in order to determine stability of the fixed point.

Example Look at $\dot{x} = x - x^3 = f(x)$, fixed points for this satisfy $f(x^*) = 0$, from which it follows that $x^*\left(1 - {x^*}^2\right) = 0 \Rightarrow x^* = 0,\ \pm 1$. In order to find stability of fixed points look at $f'(x) = 1 - 3x^2$. For $x^* = 0$, $f'(x^*) = 1$ so this is an unstable fixed point. At $x^* = \pm 1$, $f'(x^*) = 1 - 3 = -2$, so these are stable fix points. The potential U is related to $\dot{x}$ by $\dot{x} = -\frac{dU}{dx} \Rightarrow \frac{dU}{dx} = -f(x) \Rightarrow U(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + c$. Setting $c = 0$ we can plot this potential, the minima of the potential will be at $x = \pm 1$, corresponding to stable fixed points, the local maximum at $x = 0$ is an unstable fix point.

2.3 Phase Space

Lets again start with the equation of motion for a pendulum

$$ml\ddot{\theta} = -mg\sin\theta - bl\dot{\theta}$$

if we again rescale time using $\tau = \frac{t}{T}$, $T = \sqrt{\frac{l}{g}}$ and introducing $\gamma = \frac{b}{m}\sqrt{\frac{l}{g}}$ we can rewrite this as

$$\frac{d^2\theta}{d\tau^2} + \gamma\frac{d\theta}{d\tau} + \sin\theta = 0 \tag{5}$$

as it is easier to deal with a set of one dimensional differential equations we introduce $\Omega = \frac{d\theta}{d\tau}$, so we obtain from equation 5 the following set of first order differential equations

$$\begin{aligned} \frac{d\theta}{d\tau} &= \Omega \\ \frac{d\Omega}{d\tau} &= -\gamma\Omega - \sin\theta \end{aligned} \tag{6}$$

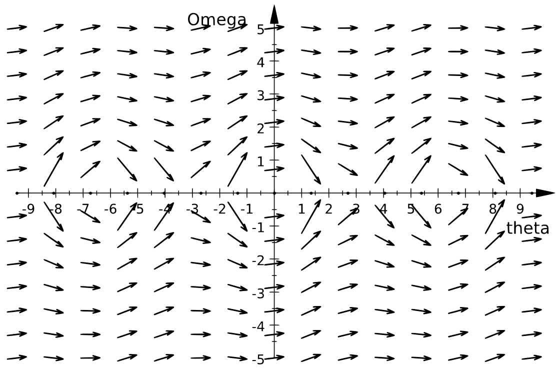

Figure 3: This is the direction field of a pendulum for small γ, in the plot $\gamma = 0.1$.

γ is often referred to as a *control parameter*, it is dimensionless and involves all interesting parameters of the system one mite want to vary. We now follow our three step recipe to solving these problems

1. find fixed points. They are given by $\frac{d\Omega}{d\tau} = 0$; $\frac{d\theta}{d\tau} = 0$, so using equation 6

$$\begin{aligned} -\gamma\Omega - \sin\theta &= 0 \\ \Omega &= 0 \end{aligned}$$

it follows straightforwardly that

$$\begin{aligned} \Omega^* &= 0 \\ \theta^* &= n\pi,\ n = 0, \pm 1, \pm 2, ... \end{aligned}$$

are the fixed points of this system.

2. plot the direction field(local direction of flow). Figure 3 shows such a direction filed for our pendulum. This plot shows $\frac{d\Omega}{d\theta}$ at several locations. It tells you in which direction the pendulum will move at a given location.

$$\frac{d\Omega}{d\theta} = \frac{d\Omega/d\tau}{d\theta/d\tau} = \frac{-\gamma\Omega - \sin\theta}{\Omega} = -\gamma - \frac{\sin\theta}{\Omega}$$

gives the gradient at each point of the graph. What happens when $\gamma = 0$? Using equation 5 we can obtain $\frac{d^2\theta}{d\tau^2} + \sin\theta = 0$. Remembering $\frac{d^2\theta}{d\tau^2} = -\frac{dU}{d\theta}$ with $U = 1 - \cos\theta$, we can rewrite this as the following(it does not matter in which order we apply differentials)

$$\begin{aligned} \frac{d\theta}{d\tau}\frac{d^2\theta}{d\tau^2} &= -\frac{dU}{d\theta}\frac{d\theta}{d\tau} \\ \frac{d}{d\tau}\left\{\frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 + U(\theta)\right\} &= 0 \end{aligned}$$

if integrated we arrive at $\frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 + U(\theta) = const = E$. Rearranging this and substituting back for U we arrive at $\frac{d\theta}{d\tau} = \pm\sqrt{2(E + \cos\theta - 1)}$. This equation can in principle be integrated by using elliptical functions but this does not tell us anything interesting about the system. Using our definition of Ωwe can write $\Omega = \pm\sqrt{2\left(E + \cos\theta - 1\right)}$, if we now also limit ourselves to small amplitudes(E small) then $\sin\theta \approx \theta$, which allows us to write[1] $\Omega^2 + \theta^2 = 2E$. In our 2D phase space this is just the equation of a circle. This means that at low values of E the pendulum will keep swinging as we would expect from classical mechanics. However it is clear that for large values of E this will not be true anymore. Its energy will be sufficient to allow it to reach the top($\theta = \pi$) and it will swing back down on the other side. As we set the friction to zero it will continue doing so for ever.

3. Linearise around the fixed points in order to perform linear stability analysis. This will tell us if a fixed point is stable or unstable.

Insert missing bits.

2.4 Non-autonomous systems

18.10.2006

Force pendulum is a non-autonomous system.

$$ml\frac{d^2\theta}{dt^2} + bl\frac{d\theta}{dt} + mg\sin\theta = F(t)$$

this looks rather complicated but can be treated as a 3D problem. Again getting rid of all units, we will write $\tau = \frac{t}{T}$, $T = \sqrt{\frac{l}{g}}$, which gives

$$\frac{d^2\theta}{d\tau^2} + \gamma\frac{d\theta}{d\tau} + \sin\theta = G\cos\left(\omega_D t\right)$$

where $\gamma = \frac{b}{m}\sqrt{\frac{l}{g}}$ and $\omega_D = \omega T = \omega\sqrt{\frac{l}{g}}$ where ω is the actual driving frequency. This system has three parameters(all dimensionless), γ is friction, G is amplitude of forcing and ω is driving frequency.

Going over to the first-order formalism

$$\begin{aligned} \frac{d\theta}{d\tau} &= \Omega \\ \frac{d\Omega}{d\tau} &= -\gamma\Omega - \sin\theta + G\cos\left(\omega_D\tau\right) \end{aligned}$$

which has explicit time dependence on right-hand side, this is the reason such systems are called non-autonomous. If we now look at

$$\frac{d\theta}{d\Omega} = \frac{\Omega}{-\gamma\Omega - \sin\theta + G\cos\left(\omega\tau\right)}$$

but as we have time in this we are in a bit of a tight spot. The slope at a given point can change with time, basically it can have any value. Promoting time to an extra dimension will allow us to get out of this tight spot. If the system had

[1] use $\cos^2 x + \sin^2 x = 1$

n dimensions before time will now become dimension $n+1$. Our system will now look like

$$\begin{aligned}\frac{dx_1}{dt} &= f_1\left(x_1, x_2, ..., x_{n+1}\right) \\ \vdots \quad & \quad \vdots \\ \frac{dx_{n+1}}{dt} &= f_{n+1}(x_1, x_2, ..., x_{n+1})\end{aligned}$$

where the last equation(x_{n+1}) is usually very simple or =1. 20.10.2006

In our case, let $\phi = \omega_D \tau$ be our new dimension, our set of equations now is

$$\begin{aligned}\frac{d\theta}{d\tau} &= \Omega \\ \frac{d\Omega}{d\tau} &= -\gamma\Omega - \sin\theta + G\cos\phi \\ \frac{d\phi}{d\tau} &= \omega_D\end{aligned}$$

we only know one way of dealing with this kind of problem so far, numerical simulation. We choose $\omega_D = \frac{2}{3}$, $\gamma = \frac{1}{2}$ and will vary G. As we are only interested in the long time behaviour of the system the initial conditions are not that important, we choose $\theta_0 = \frac{\pi}{2}$ and $\Omega_0 = 0$.

We look at plots(in phase and normal space) with various values of G. All values of G are as used by Baker&Gollub, so go there to look at pretty pictures. Values looked at: $G = 0,\ 0.13,\ 0.5,\ ....$ After this he introduced Poincare sections. We look at phase space with a period of ω_D, for *normal* systems this is rather boring and we get just one dot, but for chaotic systems it helps a lot.

2.5 Limit cycles

Limit cycles are *closed* isolated trajectories. Isolated means that neighbouring trajectories either fall back into the limit cycle(cycle is *stable*) or move away(cycle is *unstable*). Limit cycles are inherently nonlinear, they cannot occur in linear systems. Compare to the cycles found in a simple harmonic oscillator. Cycles for SHO depend on the value of E, where as for non linear systems they do not.

Artificial Example of a limit cycle. We will use the problem given by this rather constructed set of equations

$$\begin{aligned}\dot{r} &= r\left(1 - r^2\right) \\ \dot{\theta} &= 1\end{aligned}$$

there is a limit cycle for $r = 1$, which can be found by inspection. In order to determine stability of this cycle we can write $r(t) = r^* + \hat{r}(t)$, with solutions $\hat{r}(t)\,\hat{r}(0)\,e^{\alpha t}$ where $\alpha = f'(r^*)$. This gives for

$$\begin{aligned}f'(r) &= 1 - 3r^2;\ r^* = 1 \\ \Rightarrow f'(r^*) &= 1 - 3 = -2\end{aligned}$$

from which it follows that $r = 1$ is a stable limit cycle. Final state is given by $x(t) = \cos(t + \theta_0)\,,\ y(t) = \sin(t + \theta_0)$.

Poincaré oscillator is a slightly less constructed example. Problem given by

$$\begin{aligned}\dot{x} &= -b\left(\sqrt{x^2+y^2}-a\right)-\omega y\\ \dot{y} &= -b\left(\sqrt{x^2+y^2}-a\right)+\omega x\end{aligned}$$

where a,b are constants. Polar coordinates are much more useful than $x,\ y$ so introduce $r(t)$ and $\theta(t)$ via $x(t)=r(t)\cos(\theta(t))$ and $y(t)=r(t)\sin(\theta(t))$, then $r^2=x^2+y^2\Rightarrow 2r\frac{dr}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{t}$, which gives

$$\begin{aligned}r\dot{r} &= x\dot{x}+y\dot{y}\\ \vdots &= \vdots\\ &= -br(\cos\theta+\sin\theta)(r-a)\\ \dot{r} &= -b(\cos\theta+\sin\theta)(r-a)\end{aligned}$$

so for $\sin\theta\neq 0$ $r\theta=r\omega-b(\cos\dot{\theta}-\sin\theta)(r-a)$. If $r=a$, then $\dot{r}=0,\ \dot{\theta}=\omega$, so the limit cycle in this example is given by

rather messy, tedious, boring algebra

$$\begin{aligned}x(t) &= a\cos(\omega t+\theta_0)\\ y(t) &= a\sin(\omega t+\theta_0)\end{aligned}$$

Van der Pol oscillator first invented to describe non-linear electric circuits. Set of equations is

$$\ddot{x}-\alpha\left(1-x^2\right)\dot{x}+\omega_0^2 x = 0$$

where this is almost like a force SHO but has a non-linear damping term. First we want to get rid of dimensions by rescaling time $\tau=\omega_0 t$, which gives us $\frac{dx}{dt}=\omega_0\frac{dx}{d\tau},\ \frac{d^2x}{dt^2}=\omega_0\frac{d^2x}{d\tau^2}$ which allows us to write

$$\frac{d^2x}{d\tau^2}-\mu\left(1-x^2\right)\frac{dx}{d\tau}+x=0,\ \mu\equiv\frac{\alpha}{\omega_0}$$

rewriting the original equation as a set of first-order equations.

$$\begin{aligned}\frac{dx}{d\tau} &= y\\ \frac{dy}{d\tau} &= \mu\left(1-x^2\right)y-x\end{aligned}$$

We can fairly easily rule out a limit cycle and as in 2D we can only have a limit cycle or a fixed point we will then be able to say which one it will be.

Gradient system is one such technique. Can think of such systems using a mechanical analogy.

$$\begin{aligned} m\ddot{x}_1 + \dot{x}_1 &= -\frac{\partial V}{\partial x_1} \\ \vdots \quad & \quad \vdots \\ m\ddot{x}_n + \dot{x}_n &= -\frac{\partial V}{\partial x_n} \end{aligned}$$

ball of mass m moving in potential V subject to damping. Think of this as

$$\dot{x} = f_1(x_1, ..., x_n)$$

up to $\dot{x}_n$. If there exists a $V(x_1, ..., x_n)$ such that

$$f_1(x_1, ..., x_n) = -\frac{\partial V}{dx_1}$$

up to x_n, then the system is a gradient system, writing it as a vector $\vec{f} = -\nabla V$. For $n = 2$ we wold have

$$\begin{aligned} \dot{x} &= f(x,\ y) \\ \dot{y} &= g(x,\ y) \end{aligned}$$

gradient is given by $f = -\frac{\partial V}{\partial x},\ g = -\frac{\partial V}{\partial y}$. So we can test if we have a gradient system with the following

$$\frac{\partial f}{\partial y} = -\frac{\partial^2 V}{\partial x \partial y} = -\frac{\partial^2 V}{\partial y \partial x} = \frac{\partial g}{\partial x}$$

for example $f = \dot{x} = \sin y,\ g = \dot{y} = x\cos y$, then there is a potential as $\frac{\partial f}{\partial y} = \cos y,\ \frac{\partial g}{\partial x} = \cos y$. 27.10.2006

How would we find V? We have $\sin y = -\frac{\partial V}{\partial x}$and $x\cos y = -\frac{\partial V}{\partial y}$, integrating these gives

$$\begin{aligned} -V &= x\sin y + \psi(y) \\ \frac{\partial V}{\partial y} &= -x\cos y - \frac{d\psi}{dy} \end{aligned}$$

by comparing this to $x\cos y = -\frac{\partial V}{\partial y}$ we can deduce that $\psi = 0$. From this it follows that $V(x,\ y) = -x\sin y + const.$

Closed orbits are impossible in gradient systems, we can prove this by supposing that there were a closed orbits in the system with a potential $V(x,\ y)$. Assuming this, after one circuit the change in V, ΔV has to be zero. But $\Delta V = \int_0^T \frac{dV}{dt}dt$ and $\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} + \frac{dV}{dy}\frac{dy}{dt} = -\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2$. From this it follows that

$$\Delta V = -\int_0^T \left\{ \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 \right\} dt \leq 0$$

with $\Delta V = 0$ only if $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 0$ for all t. If $\dot{x}$ or $\dot{y}$ are zero we are not moving anywhere in time, fixed point. It is hence impossible to have limit cycles if there is a potential V.

Liapunov functions are functions $\phi(x,\ y)$ with following properties:

- $\phi(x,\ y) > 0$ for all $x,\ y$ except at a fixed point $(x^*,\ y^*)$ where $\phi(x^*,\ y^*) = 0$.
- $\frac{d\phi}{dt} < 0$ for all $x,\ y$ apart from $(x^*,\ y^*)$
- $x(t) \to x^*,\ y(t) \to y^*$ as $t \to \infty$ for all initial conditions, it lets us talk about global stability

Can also prove that there are no limit cycles, by again noting that $\Delta\phi = 0$ for one circuit of period T and $\Delta\phi = \int_0^T \frac{d\phi}{dt} dt < 0$ unless we are at the fix point.

Example specially constructed so we can find a Liapanow function.

$$\begin{aligned} \dot{x} &= -x + 4y \\ \dot{y} &= -x - y^3 \end{aligned}$$

We can try $\phi = x^2 + ay^2$ where a is a parameter to be chosen later. Fixed points $x = 4y,\ x = -y^3 \Rightarrow 4y = -y^3 \Rightarrow y = 0$, hence only one fixed point at $(0,0)$. Sub into ϕ, $\phi(0,0) = 0\surd$ if $a > 0$ for $x \neq 0,\ y \neq 0$. This gives

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{dt}\frac{dy}{dt} = 2x(-x + 4y) + 2ay\left(-x - y^3\right) \\ &= -2x^2 + 2xy(4 - a) - 2ay^4 \end{aligned}$$

if we choose $a = 4$ we get

$$\frac{d\phi}{dt} = -2x^2 - 8y^2 < 0$$

so $\phi = x^2 + 4y^2$ is a Liapunov function for this system.

Example Look at

$$\begin{aligned} \dot{x} &= -x + 2y \\ \dot{y} &= -x - y - y^3 \end{aligned}$$

so that we immediately get

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{\partial\phi}{dt}\frac{dx}{dt} + \frac{\partial\phi}{dt}\frac{dy}{dt} \\ &= -2x^2 + 4xy - 2xy - 2y^2 - 2y^4 \\ &= -2x^2 - 2y^2 - 2xy - 2y^4 \end{aligned}$$

if $xy < 0$, then $\frac{d\phi}{dt} < 0$ but if $xy > 0$ we are in trouble. We can use an inequality for $xy > 0$, $(x - y)^2 \geq 0$ with equality when $x = y$, can expand this inequality to $2xy \leq x^2 + y^2$, substituting this back in we get

$$\frac{d\phi}{dt} \leq -2x^2 - 2y^2 + x^2 + y^2 - 2y^4 = -x^2 - y^2 - 2y^4 \leq 0$$

6.11.2006

The Poincaré-Bendixson theorem, Suppose R is a region in the plane which contains no fixed points and suppose that there is a trajectory C that is "confined" in R. Then either

1. C is a closed orbit, or,
2. C spirals towards a closed orbit as $t \to \infty$

in either case R contains a closed orbit. The standard way of applying the theorem is to construct a trapping region, in which the slope field points inwards everywhere on the boundary of R. As an example consider

$$\begin{aligned} \dot{r} &= r\left(1-r^2\right) + \mu r \cos\theta \\ \dot{\theta} &= 1 \end{aligned}$$

For $\mu = 0$, there is a limit cycle at $r = 1$, see earlier.

We want to show that there is still a limit cycle for small positive μ. Construct two circles of radius r_{min} and r_{max}, defining our trapping region. We want to choose radii so that $\dot{r} > 0$ for $r = r_{min}$ and $\dot{r} < 0$ for $r = r_{max}$.

r_{min} We require that $\left(1-r^2\right)+\mu\cos\theta > 0$ for all θ, but $\cos\theta > -1 \Rightarrow r^2 < 1-\mu$ over all θ. So take $r_{min} = 0.9999\,(1-\mu)^{\frac{1}{2}}$, then $\dot{r} > 0$ for all θ. This holds for all $0 < \mu < 1$.

r_{max} Ask that $r\left(1-r^2\right) + \mu r\cos\theta < 0$ for all θ, i.e. $\left(1-r^2\right) + \mu\cos\theta < 0$ or $r^2 > 1 + \mu\cos\theta$, but $\cos\theta < 1 \Rightarrow r^2 > 1+\mu$, from which follows that $r > (1+\mu)^{\frac{1}{2}}$, so set $r_{max} = 1.0001\sqrt{1+\mu}$.

This tells us that a closed orbit exists for $0 < \mu < 1$ and lies somewhere in the annulus $0.99\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$.

Poincaré-Bendixson theorem tells us that dynamical possibilities in the phase plane are very limited, but PB-theorem has no analogue in three, four or more dimensions and trajectories may be attracted to a completely geometrical object called a STRANGE ATTRACTOR leading to chaotic behaviour.

2.6 Chaotic dynamics - Lorenz equation

In 1963 Lorenz was studying a very simplified model of the atmosphere. He assumed surface of the earth to be at temperature T_1 and the atmosphere at a certain height to be at temperature T_2, where $T_1 > T_2$. Warm air rises, cool air descends, giving rise to convection currents which may form convection rolls. Can investigate this in the lab by putting fluid between two plates and heating from below. This system is described by five PDEs for the fluid density $\rho(\vec{x},\ t)$, temperature $T(\vec{x},\ t)$ and velocity $\vec{v}(\vec{x},\ t)$. To simplify these do a Fourier expansion.

$$\begin{aligned} T(\vec{x},\ t) &= \sum_{n,m} a_{mn}(t) \begin{pmatrix} \sin \\ \cos \end{pmatrix} \frac{n\pi z}{l} \begin{pmatrix} \sin \\ \cos \end{pmatrix} \frac{n\pi x}{l} \\ \rho(\vec{x},\ t) &= \sum_{n,m} b_{mn}(t) \begin{pmatrix} \sin \\ \cos \end{pmatrix} \frac{n\pi z}{l} \begin{pmatrix} \sin \\ \cos \end{pmatrix} \frac{n\pi x}{l} \end{aligned}$$

He threw away all nodes but the lowest three, this gives you three ordinary differential equations for three of the time-dependent amplitudes $a_{nm}(t)$, $b_{nm}(t)$, ... Let us call these amplitudes $x(t)$, $y(t)$, $z(t)$(not coordinates!), after rescaling we get

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx-y-xz\\ \dot{z} &= xy-bz\end{aligned}$$

where $\sigma,\ r,\ b$ are positive constants. These are known as the Lorenz equations.

r is a scaled form of $\Delta T = T_1 - T_2$

σ is the Prondt number

The fixed points of this system have to satisfy $x^* = y^*$ from which follows that $(x^*)^2 = bz^*$ and $x^*(r-1-z^*) = 0$. So either $x^* = 0 \Rightarrow z^* = 0$ and $y^* = 0$ or $z^* = r-1 \Rightarrow (x^*)^2 = (y^*)^2 = b(r-1)$. 8.11.2006

Fixed points are hence given by

$$\begin{gathered}(0,0,0)\\ \left(\pm\sqrt{b(r-1)},\ \pm\sqrt{b(r-1)},\ r-1\right) \quad for\ r>1\end{gathered}$$

Now we do stability analysis. For $r<1$ we can set up a Liapunov function to show that the origin is *globally* stable. Take $\phi(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2$ and check that this is a Liapunov function. We require

1. $\phi > 0$ except at the fixed point $(0,0,0)$.
2. $\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} < 0$

These are the two conditions that need to be true for it to be a Liapunov function. If we do the time derivative we get

$$\begin{aligned}\frac{1}{2}\frac{d\phi}{dt} &= xy - x^2 - y^2 + rxy - bz^2\\ &= -\left[x - \frac{r+1}{2}y\right]^2 + \frac{(r+1)^2}{4}y^2 - y^2 - bz^2\\ &= -\left[x - \frac{r+1}{2}\right] + \left[1 - \frac{(r+1)^2}{4}\right]y^2 - bz^2\end{aligned}$$

for $\frac{(r+1)^2}{4} < 1$, we have $\frac{d\phi}{dt} < 0$.

$$\begin{aligned}-1 < &\ \tfrac{r+1}{2} &< 1\\ -2 < &\ r+1 &< 2\\ -3 < &\ r &< 1\end{aligned}$$

from this it follows that ϕ is a Liapunov function as long as $r<1$. Hence the origin is globally stable and there are no limit cycles if $r<1$. If we trace back what the fix points mean, then $(0,0,0)$ represents heat transfer without convection(only conduction) and $\left(\pm\sqrt{b\,(r-1)},\ \pm\sqrt{b\,(r-1)},\ r-1\right)$ represent right or left turning convection rolls – they only exist for $r>1$. We can use a short hand of c^+ and c^- for the positive and negative fixed point.

What happens if we increase r?

- The fixed points $c^+$and c^- are stable for $r>1$ if $\sigma<b+1$, but if $\sigma>b+1$ then they are only stable for a range of r, given by $1<r<\frac{\sigma(\sigma+b+3)}{\sigma-b-1}\equiv r_1$.
- There are no limit cycles for $r<r_1$ or for values of r somewhat greater then r_1.
- All trajectories eventually enter and remain in a trapping region – so trajectories are not repelled out to infinity. Lorenz choose $\sigma=10,\ b=\frac{8}{3}$(then $\sigma>b+1$), this gives $r_1=24.74$, so he took $r=28$. There is no particular reason for why he choose these values, you can in principle choose what ever suits you as long as $\sigma>b+1$ is true.

In the full 3D phase space the attractor appears to lie on a thin surface that looks like a pair of butterfly wings. Actually the "surface" is an infinite complex of surfaces-a set of points with infinite surface area, but zero volume. It is called a STRANGE ATTRACTOR and has fractal structure.

Exponential divergence of nearby trajectories Two trajectories which start very close to each other on the attractor may rapidly diverge from each other and have very different futures. This means the systems exhibit sensitive dependence on initial conditions.

Suppose $\vec{x}\,(t)$ is a point on the attractor at time t and $\vec{x}\,(t)\,\delta\vec{x}\,(t)$ is a nearby point(say $|\delta\vec{x}|=10^{-10}$). Numerically for the Lorenz attractor it is found that

$$|\delta\vec{x}\,(t)|=|\delta\vec{x}\,(0)|\exp\left(Lt\right)$$

where $L=0.9$. We call L a Liapunov exponent.

If our prediction ceases to be tolerable when $|\delta\vec{x}\,(t)|\gtrsim a$ then this will happen after a time $t\approx\frac{1}{L}\ln\frac{a}{|\delta\vec{x}(0)|}$. This holds for all chaotic systems, systems for which $L>0$.

Chaos is a periodic long term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions.

System does not settle down to a fixed point, limit cycle, ... There are no random inputs or noise.

3 The Logistic Map

3.1 Linear and Quadratic maps

13.11.2006

A map is a dynamical system in which time is discrete, e.g. $x_{n+1}=\cos x_n$ where $n=1,2,3,...$, if we know n_0 we can work out every x. An example of a linear map is $f\,(x)=\alpha x+\beta$, this can be solved exactly and hence is rather boring.

The next more complicated map is a quadratic map $f(x) = \alpha x^2 + \beta x + \gamma$. Obviously we can choose α and β so that we need not keep γ. We can rescale and get rid of constants and write the equation in more common form as

$$z_{n+1} = \lambda z_n (1 - z_n)$$

we could just write x instead of z. It is also convention to use $0 < x_0 < 1$. We would also like x in general be $0 < x < 1$, this requires $0 < \lambda < 4$.

It's most convenient to study return maps which map a finite interval to itself.

3.2 Simple analysis of the logistic map

The maximum value of the map given by $x_{n+1} = \lambda x_n (1 - x_n)$ which is $\frac{\lambda}{4}$, its is also symmetric about $x = \frac{1}{2}$. Fixed points will be when there is no difference between steps anymore, $x_{n+1} = x_n = x^*$. Obviously $x^* = 0$ is always a solution but if we require $x^* \neq 0$ then we have $1 = \lambda (1 - x^*) \Rightarrow x^* = 1 - \frac{1}{\lambda}$.

Lets look at what happens for $\lambda < 1$. There is only one fixed point, at $x = 0$. We have $f(x) = \lambda x - \lambda x^2$, this is always less than λx. In order to test for stability we look at a Cobweb construction. Move horizontally from value of x_1 on the graph to the diagonal line, vertically to the curve again, horizontally to the diagonally, etc.

What happens for $\lambda > 1$? We now have two fixed points, one at $x = 0$ and one at $x = 1 - \frac{1}{\lambda} < 1$. It now appears that $x^* = 1 - \frac{1}{\lambda}$ is stable but $x^* = 0$ is unstable. When λ gets close to 4 there are no fixed points, we get what looks like random behaviour.

We can also do the analog of linear stability analysis. Write $x_n = x^* + \hat{x}_n$, $x_{n+1} = f(x_n)$ so $x^* + \hat{x}_{n+1} = f(x^* + \hat{x_n}) = f(x^*) + \hat{x}_n f'(x^*) + O()$, since $x^* = f(x^*)$ we can write $\hat{x}_{n+1} = f'(x^*) \hat{x_n}$. We have stability when $\hat{x}_{n+1}$is smaller in magnitude then $\hat{x}_n$. Generally

$$\begin{aligned} |f\prime(x^*)| &\quad < 1 \quad \rightarrow \textit{stable} \\ |f\prime(x^*)| &\quad > 1 \quad \rightarrow \textit{unstable} \end{aligned}$$

For logistic map we have $\frac{df}{dx} = \lambda - 2\lambda x$, so for $x^* = 0$ we have $\frac{df}{dx} = \lambda$, stable for $\lambda < 1$, unstable for $\lambda > 1$.

For $x^* = 1 - \frac{1}{\lambda}$ we have $\frac{df}{dx}|_{x=x^*} = \lambda - 2\lambda \left(1 - \frac{1}{\lambda}\right) = 2 - \lambda$. This point exists if $\lambda > 1$ and is stable if $\lambda < 3$, unstable if $\lambda > 3$.

What happens if $\lambda > 3$ when both fixed points are unstable? If we do this numerically we will find that the iterates settle down to oscillate between two numbers p and q, which only depend on λ. that is $q = f(p)$ and $p = f(q)$. Think about applying f twice, $f(f(p)) = f(q) = p$ and $f(f(q)) = f(p) = q$. 15.11.2006

So we can treat these like fixed points of the function $g(x) = f(f(x))$.

$$\begin{aligned} f(f(x)) &= \lambda f(x)(1 - f(x)) \\ &= \lambda^2 x (1 - x)[1 - \lambda x (1 - x)] \end{aligned}$$

so for fixed points we need to solve

$$\begin{aligned} x = & \quad \lambda^2 x (1-x) [1 - \lambda x (1-x)] \\ \Rightarrow \quad x & \left\{ \lambda^2 - \lambda^2 x - \lambda^3 x (1-x)^2 - 1 \right\} \quad = 0 \end{aligned}$$

knowing the fixed points and that $f(x^*) = x^*$ and $f(f(x^*)) = f(x^*) = x^*$ we can factorise this to

$$\underbrace{x(\lambda x + 1 - \lambda)}_{x^* = 0,\, x^* = 1 - \frac{1}{\lambda}} \quad \underbrace{\left(-\lambda^2 x^2 + \lambda(\lambda+1) x - (\lambda+1)\right)}_{p \ and \ q} \quad = 0$$

using p-q or abc formula we can find p and q to be

$$p, q = \frac{\lambda+1}{2\lambda} \pm \frac{(\lambda+1)^{\frac{1}{2}} (\lambda-3)^{\frac{1}{2}}}{2\lambda}$$

Since x^*, p, q are fixed points of $f(f(x))$ call this function $g(x)$ and focus on this $g(x) = \lambda^2 x (1-x) [1 - \lambda x (1-x)]$.

Stability of the 2-cycle? Look at, steps in the 2 cycle,

$$x_n = x_n^* + \hat{x}_n, \; x_{n+2} = x^* + \hat{x}_{n+2}$$

we can compare the two hat values and telling us if we move towards or away from the fixed point.

We have $\hat{x}_{n+2} = g\prime(x^*)\hat{x}_n$, x^* is stable if $\left|\frac{dg}{dx}\right|_{x=x^*} < 1$ and unstable if $\left|\frac{dg}{dx}\right|_{x=x^*} > 1$. But we have $\left.\frac{dg}{dx}\right|_{x=p} = f\prime(f(p)) f\prime(p) = f\prime(q) f\prime(p)$ and similarly $\left.\frac{dg}{dx}\right|_{x=q} = f\prime(f(q)) f\prime(q) = f\prime(p) f\prime(q)$. This means we only have to work out $f\prime(x)$ once and then look at the result.

If we do this for our example we get

$$\begin{aligned} f\prime(p) &= -1 - (\lambda+1)^{\frac{1}{2}} (\lambda-3)^{\frac{1}{2}} \\ f\prime(q) &= -1 + (\lambda+1)^{\frac{1}{2}} (\lambda-3)^{\frac{1}{2}} \end{aligned}$$

which gives $f\prime(p) f\prime(q) = 4 + 2\lambda - \lambda^2$. This means that 2-cycle will be stable if $\left|4 + 2\lambda - \lambda^2\right| < 1$. We can rewrite this as

$$\begin{aligned} -1 &< \lambda^2 - 2\lambda - 4 < 1 \\ 0 &< \lambda^2 - 2\lambda - 3 < 2 \end{aligned}$$

which gives $\lambda^2 - 2\lambda - 3 > 0$ and $\lambda^2 - 2\lambda - 5 > 0$. We can find the roots of these equations/solve the inequalities. They are -1 and -3 for first equation and $1 \pm \sqrt{6}$ for second equation. This means that 2-cycle exists and is stable for

$$3 < \lambda < 1 + \sqrt{6} \approx 3.44$$

this means that for $\lambda > 3$ we have a 2–cycle and for $\lambda > 1 + \sqrt{6}$ we would have a 4–cycle.

3.3 The Numerical analysis of logistic map

Just beyond $\lambda = 1 + \sqrt{6}$, both fixed points are unstable, as are p and q – the stable long time behaviour consists of jumping between four numbers $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}$. They are fixed points of $h(x) = f(f(f(f(x))))$. As λ is increased still further find a succession of attractors of period 8,16,...,2^l,... Let Λ_l be the value of λ for which the period 2^l–cycle first appears, eg $\Lambda_1 = 3$(period 2–cycle born), $\Lambda_2 = 1 + \sqrt{6}$(period 4–cycle born), $\Lambda_3 = 3.544$(period 8-cycle born), etc.

Feigenbaum constructed the ratio

$$\delta_n \equiv \frac{\Lambda_{n+1} - \Lambda_n}{\Lambda_{n+2} - \Lambda_{n+1}}$$

if we do this for our example we will find that $\delta_1 = 4.751$, $\delta_2 = 4.657$, $\delta_3 = 4.664$, ..., if we continue doing this δ will approach a constant, now called δ_1 as n becomes large

$$\delta = \lim_{n\to\infty} \frac{\Lambda_{n+1} - \Lambda_n}{\Lambda_{n+2} - \Lambda_{n+1}} = 4.699201609...$$

no matter what system we have we always find this number as the value of δ. 17.11.2006

Feigenbaum showed

$$\begin{aligned} \Lambda_\infty - \Lambda_n &= \sum_{i=0} (\Lambda_{n+i+1} - \Lambda_{n+i}) \\ &= \frac{1}{1 - 1/\delta} (\Lambda_{n+1} - \Lambda_n) \end{aligned}$$

which eventually leads to $\Lambda_\infty - \Lambda_n \approx C\delta^{-n}$, so we can find δ by taking the log of both sides and plotting it.

We find δ is universal, it is independent of the detailed form of $f(x)$.

What happens after $\Lambda_\infty = 3.5699...$ when an attractor of infinite periodicity appears? For $\Lambda_\infty < \lambda < 4$, a mixture of chaos and order is found – chaos, but with "windows" of periodic behaviour. There is a large window starting at $\lambda = 1 + 2\sqrt{2} \approx 3.82...$, begins with a period 3-cycle, period doubles to 6, to 12,.... Similar behaviour in other nonlinear maps, e.g. sine map $x_{n+1} = \lambda \sin(\pi x_n)$, $0 < \lambda < 1$, $0 < x < 1$.

Finding the Liapunov exponent for the logistic map. We start with two nearby points:x_0 and $x_0 + \delta x_0$. Now apply the map n times(go through n time steps), this will give x_n and $x_n + \delta x_n$, or

$$\begin{aligned} x_n &= f(f(f(f(...f(x_0))))) = f^n(x_0) \\ x_n + \delta x_n &= f^n(x_0 + \delta x_0) \end{aligned}$$

If we have $|\delta x_n| = |\delta x_0| e^{Ln}$, then L is the Liapunov exponent, if $L > 0$ then we have chaos. Find $L = \frac{1}{n} \ln \frac{\delta x_n}{\delta x_0} = \frac{1}{n} \ln \left(\frac{f^n(x_0+\delta x_0) - f^n(x_0)}{\delta x_0} \right)$ if we let $\delta x_0 \to 0$we get

$$L = \frac{1}{n} \ln \left(\left. \frac{df^n}{dx} \right|_{x=x_0} \right)$$

How do we work this out? Consider $\frac{d}{dx}f^2(x) = \frac{d}{dx}f(f(x)) = f\prime(f(x))\,f\prime(x)$ from which follows that $\left.\frac{d}{dx}f^2(x)\right|_{x=x_0} = f\prime(x_1)\,f'(x_0)$. This holds true in general and gives

$$\begin{aligned}\left.\frac{df^n}{dx}\right|_{x=x_0} &= f\prime(x_{n-1})\,f\prime(x_{n-2})\ldots f\prime(x_1)\,f\prime(x_0)\\ &= \prod_{i=0}^{n-1} f\prime(x_i)\\ \Rightarrow \ln\left|\frac{df^n}{dx}\right|_{x=x_0} &= \sum_{i=0}^{n-1}\ln|f\prime(x_i)|\end{aligned}$$

so we get $L = \frac{1}{n}\sum\ln(f\prime(x_i))$. Consider a fixed point x_0, then we have $x_1 = x_0 = x_2$, this gives $L = \frac{1}{n}n\ln f\prime(x_0) = \ln(f\prime(x_0))$ but we found earlier $|f\prime(x_0)| < 1$, also implied by the fact its a fixed point, so $L < 0$. No chaos.

We could plot L as a function of λ, this should gives us a good picture of where the "windows" are. Every time we get $L = 0$ we have found a fixed point, when $L > 0$ we are in a chaotic region and when $L < 0$ we are in a "window".

3.4 Binary shift map

29.11.2006

Consider the logistic map

$$\begin{aligned}x_{n+1} &= \lambda x_n(1-x_n)\\ &= 4x_n(1-x_n)\end{aligned}$$

with $\lambda = 4$. Now let $x_n = \sin^2(\pi y_n)$ for all n. The map now becomes

$$\begin{aligned}x_{n+1} &= 4\sin^2(\pi y_n)\cos^2(\pi y_n)\\ &= \sin^2(2\pi y_n)\end{aligned}$$

we now also have $x_{n+1} = \sin^2(\pi y_{n+1}) = \sin^2(2\pi y_n)$. If we start by looking at $0 \le y_n < \frac{1}{2}$, then $y_{n+1} = 2y_n$ lies in the interval $(0,1)$, but if $\frac{1}{2} \le y_n < 1$, then $y_{n+1} = 2y_n$ does not lie in interval $(0,1)$. However $y_{n+1} = 2y_n - 1$ does lie in $(0,1)$ and also satisfies $\sin^2(\pi y_{n+1}) = \sin^2(2\pi y_n)$. Show this!

So an equivalent map to the logistic map when $\lambda = 4$ is

$$y_{n+1} = \begin{cases}2y_n & if\ 0 \le y_n < \frac{1}{2}\\ 2y_n - 1 & if\ \frac{1}{2} \le y_n < 1\end{cases}$$

this maps the unit interval to itself repeatedly.

There is only one fixed point at $y^* = 0$, we can see this if we plot the map and then the diagonal. We can also show it analytically though,

$$y^* = \begin{cases}2y^* & 0 \le y^* < \frac{1}{2}\\ 2y^* - 1 & \frac{1}{2} \le y^* < 1\end{cases}$$

which leads to $y^* = 0$ as the only fixed point.

Stability of this map, look at derivative of $y_{n+1} = f(y_n)$, where f is our map. We find that $f\prime(y) = 2$, this means that $y^* = 0$ is unstable as $|f\prime(y^*)| > 1$ always.

What about $f(f(y))$? If $0 \leq y < \frac{1}{4}$, then $f(f(y)) = 4y$. If $\frac{1}{4} \leq y < \frac{1}{2}$, then $f(f(y)) = 4y - 1$. if $\frac{1}{2} \leq y < \frac{3}{4}$, then $f(f(y)) = 2(2y-1) = 4y - 2$ and if $\frac{3}{4} \leq y < 1$then $f(f(y)) = 4y - 3$. A neater way to write this is $f(y) = 2y\,(mod\ 1)$ and then we get $f(f(y)) = 4y\,(mod\ 1)$. Similarly we get $f(f(f(<))) = 8y\,(mod\ 1)$.

All fixed points of this map is unstable, how to show that? $f\prime = 2$, $\frac{d}{dy}f(f(y)) = 4$ and in general we get $\frac{d}{dy}f^{(n)}(y) = 2^n$.

Can we find a two-cycle? They satisfy $y^* = f(f(y^*))$, we find that $y^* = \frac{1}{3}$ and $y^* = \frac{2}{3}$ to be two cycles. How? We can do it pictorially by plotting y_n vs y_{n+2} and drawing a diagonal and see where they cross.

All this generalizes to $f(x) = bx\,(mod\ 1)$ with $b = 3, 4, 5, ...$ Why is the $b = 2$ map called the binary shift map? To see this we look at $b = 10$, called the decimal shift map,

$$x_{n+1} = 10x_n\,(mod\ 1)$$

but with x_n expressed as a decimal, not a fraction. If we have $x_n = 0.5642$ then $x_{n+1} = 0.642$. All digits have shifted left by one place and the first one has be deleted. This is true for any b, as long as the number is expressed as a decimal to the base b.

What about $b = 2$, 17 is 16+1=$2^4 + 2^0 = 10001$ or $\frac{13}{16} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} = 0.1101$ in binary. The fixed point s of this map will be $x^* = 0.0000...,\ 0.111111.....$, we 1.12.2006
only have 0 and 1 to play with. In decimal the fixed points are $x^* = 0,\ 1$.

Rational numbers, fractions are either terminating or repeating decimals. Look at these numbers

$$\begin{aligned}
\frac{7}{8} &= 0.875 \\
\frac{3}{7} &= 0.\underline{428571428571}4285 \\
\frac{9}{55} &= 0.1\underline{636363}6363636364
\end{aligned}$$

conversely any recurring decimal is a rational number. On the other hand irrational numbers do not terminate or repeat, when expressed as decimals.

So if we start with x_0 being a rational number, only two things can happen when the following map is applied

$$x_{n+1} = bx_n\,(mod\ 1)$$

either a fixed point(it trites) or we end up in a cycle(it repeats itself). The cycle will be unstable.

If the starting value is irrational x_n will never settle down to a fixed, periodic behaviour, the motion is a periodic.

Trying to find the Liapunov exponent, suppose x_0 and $x\prime_0$ are two starting values, then

$$\begin{aligned}
f^n(x_0) - f^n(x\prime_0) &= b^n x_0 - b^n x\prime_0 \\
x_n - x'_n &= b^n(x_0 - x'_0) \\
\Rightarrow \delta x_n &= e^{Ln}\delta_{x_0}
\end{aligned}$$

so we get $e^{Ln} = b^n \Rightarrow b = e^L$ or $L = \ln b$. So for the binary shift map, the Liapunov exponent is $L = \ln 2 > 0$.

4 Fractals

4.1 Introduction

How long is the coastline of Britain? This sounds trivial but is actually a bit more complicated. Take a map(EGA on a scale 1:1,000,000) and set a compass at a fixed width and see how many "steps" one can make going around the coast. 6.12.2006

Let's define some terms and symbols. Let η be the compass setting(the length we are using to measure the coastline). Call the length of the coastline $L(\eta)$. If $L(\eta)$ is plotted against η on a log-log scale we find a constant slope

$$\ln L = \alpha \ln \eta + \ln C$$

from which follows $L(\eta) = C\eta^\alpha$, where C,α are constants. For the coast of Britain $\alpha = -0.36$, where for a straight line $\alpha = 0$.

The coast of Britain is a fractal object with complex structure on all scales and which has some degree of self-similarity.

To get some insight let's look at an artificially constructed fractal, the VON KOCH CURVE. Begin with a straight line segment, S_0, of length l. Split it into three parts, take out middle bit, replace it by a "triangle". Now there are for pieces to this line, each $\frac{1}{3}l$ long. We continue doing this. Split every line into three parts, and replace the middle one with two lines. Denote the line obtained after the first replacement with S_1, after second replacement we call it S_2 and so on. The length of S_3 is given by $\left(\frac{4}{3}\right)^3 l$, S_4 has length $\left(\frac{4}{3}\right)^4 l$, so S_n has length $\left(\frac{4}{3}\right)^n l$. The VON KOCH CURVE S_k, where $k = \infty$, clearly has infinite length. The VON KOCH CURVE is an artificial fractal; it has exactly self-similarity and structure down to infinitely fine scales.

4.2 Similarity dimension

Take a square, S_0, and divide it into four little ones, S_1, that exactly fill the first square S_0. We scaled the square down by a factor of two, $r = 2$, and four copies of S_1 are needed to cover the original, $m = 4$. We see that for a square we get the relationship $m = r^2$.

If we did this whole thing for a cube(three dimensional object), we get $m = r^3$. Similarly for a hypercube in d-dimensions, $m = r^d$. From this we take that the dimension is given by d. Given by $\ln m = d \ln r \Rightarrow d = \frac{\ln m}{\ln r}$.

Let's now do this for the VON KOCH CURVE. S_1 is made up of four equal pieces, each of which is similar to S_0 but scaled down by a factor of three. This implies that $r = 3$ but $m = 4$, we created four copies. This gives

$$d = \frac{\ln 4}{\ln 3} = 1.261...$$

a non-integer dimension. 8.12.2006

How is d related to α–the exponent appearing in $L(\eta)$. Well, $\eta = \frac{l}{r^n}$ is the scale relevant to the n-th segment S_n.

$$\begin{aligned} r^{-n} &= \frac{\eta}{l} \\ -n \ln r &= \ln \frac{\eta}{l} \\ n &= -\frac{\ln \frac{\eta}{l}}{\ln r} \end{aligned}$$

but we have $L_n = \left(\frac{m}{r}\right)^n l \Rightarrow \ln \frac{L}{l} = n \ln \frac{m}{r}$. If we now combine those two equations to eliminate n

$$\begin{aligned} \ln \frac{L}{l} &= \left[\frac{\ln \frac{\eta}{l}}{\ln r}\right] \ln \frac{m}{r} \\ &= -\ln \frac{\eta}{l} \left[\frac{\ln m - \ln r}{\ln r}\right] \\ &= -\ln \frac{\eta}{l} [d-1] \\ \frac{L}{l} &= \left(\frac{\eta}{l}\right)^{-(d-1)} \end{aligned}$$

this gives $L = \frac{l\eta^{1-d}}{l^{1-d}}$, comparing this to $L(\eta) = C\eta^\alpha$ we find $\alpha = 1 - d$ and $C = l^d$.

Some other self-similar fractals:

- The Cantor set, start with s_0, the interval $[0, 1]$. Remove its middle third, ie the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$[2]. Next time you take out middle interval from the two intervals you got before. S_1 is composed of two copies of S_0 $(m = 2)$ each scales down by a factor of three $(r = 3)$, this gives its dimension as $d = \frac{\ln 2}{\ln 3} = 0.63...$
- The von Koch snowflake, or von Koch island. This is basically a closed curve version of the von Koch curve. The snowflake S_∞ has finite area but infinite perimeter.
- Sierpinski triangle and sponge
- Menger Sponge

5 Further aspects of chaotic dynamics

5.1 How do volumes evolve in phase space

See handout.

5.2 Folding and stretching

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Trajectories do not cross, lie on attractors of zero volume as $t \to \infty$(for non-conservative systems), separate exponentially fast from their neighbors(at least

[2] [...]closed interval(includes end points), (...) open interval(does not include end points)

initially). If they remain confined to a bounded region of phase space they are strange attractors.

How can trajectories on strange attractors have all these properties?

Making pastry imagine a bit of dough which gets flattened and stretched into a plane, then folded over into a "horseshoe" like object, which then again gets stretched and flattened. If we keep on with this construction, eventually we get S_∞ which has infinitely many layers separated by gaps. A vertical cross-section through the middle of S_∞ would resemble a Cantor set.

The Bakers' map The Bakers' map, B, of the square $0 \leq x < 1,\ 0 \leq y < 1$ to itself is given by

$$(x_{n+1},\ y_{n+1}) \begin{cases} (2x_n,\ ay_n) & if\ \ 0 \leq x_n < \frac{1}{2} \\ \left(2x_n - 1,\ ay_n + \frac{1}{2}\right) & if\ \ \frac{1}{2} \leq x_n < 1 \end{cases}$$

where a is a parameter in the range $0 < a \leq \frac{1}{2}$. That is $(x_{n+1},\ y_{n+1}) = B\left(x_n,\ y_n\right)$where

$$B\left(x,\ y\right) \begin{cases} (2x,\ ay) & if\ \ 0 \leq x < \frac{1}{2} \\ \left(2x - 1,\ ay + \frac{1}{2}\right) & if\ \ \frac{1}{2} \leq x < 1 \end{cases}$$

If $0 \leq x < \frac{1}{2}$, then $B\left(x, y\right) = (2x,\ ay)$, this is the flatten and stretch bit. If $\frac{1}{2} \leq x < 1$, then $B\left(x, y\right) = \left(2x - 1,\ ay + \frac{1}{2}\right)$ as for $0 \leq x < \frac{1}{2}$ but now move up by $\frac{1}{2}$ and place over the interval $0 \leq x < \frac{1}{2}$.

The map $B^n\left(s\right)$ consists of 2^n stripes each of height a^n. In the limit $n \to \infty$, we have a Contor set of the segments. The total height of the shaded areas is $2^n \times a^n = (2a)^n$.